Fields Institute Communications
Volume **00**, 0000

On the Multiplication in the Quantized Enveloping Algebra of Type A

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This communication is dedicated to Vlasta Dlab.

1 Introduction

This is joint work with Robert Bédard. We will describe a formula for the multiplication of a root vector and an arbitrary element of a PBW-basis in the quantized enveloping algebra of type A. The initial motivation for this problem was the study of rational smoothness of the closure of orbits of representations of quivers, and there are interesting applications of the formula in this context, cf. [BS] and [Sch]. Root vectors have been considered before in [Ros89] and [Béd95]. The author thanks the referee for helpful comments on the presentation.

Let F be an algebraically closed field of characteristic > 0. Let v be an indeterminate and $\mathbf{U}$ be the Drinfeld-Jimbo quantized enveloping algebra over $\mathbf{Q}(v)$ of type A_n. Let $\mathbf{U}^+$ be the positive part of $\mathbf{U}$. $\mathbf{U}^+$ is a $\mathbf{Q}(v)$-algebra with generators: $E_1, \ldots, E_n$ and relations:

$$\begin{cases} E_i^2 E_j - (v+v^{-1}) E_i E_j E_i + E_j E_i^2 = 0, & \text{if } |i-j| = 1 \\ E_i E_j - E_j E_i = 0, & \text{if } |i-j| > 1. \end{cases}$$

Let $(^-) : \mathbf{U}^+ \to \mathbf{U}^+$ be the $\mathbf{Q}$-algebra involution defined by $\overline{E_i} = E_i$, and $\overline{v} = v^{-1}$.

Let R, (R^+) be the set of (positive) roots of a root system of type A_n. Denote by $\alpha_1, \ldots, \alpha_n$ the simple roots, then we have $R^+ = \{(\alpha_a + \alpha_{a+1} + \ldots + \alpha_b) \mid 1 \leq a \leq b \leq n\}$. Let ν be the number of positive roots, $\nu = n(n+1)/2$. Any $\alpha \in R$ defines a reflection s_α in the Weyl group. We will write s_i instead of s_{α_i}. Let w_0 be the longest element in the Weyl group.

Given an integer $N \geq 0$, we define

$$[N]! = \prod_{h=1}^{N} \frac{(v^h - v^{-h})}{(v - v^{-1})} \text{ and } E_i^{(N)} = \frac{E_i^N}{[N]!} \text{ for } 1 \leq i \leq n.$$

1991 *Mathematics Subject Classification.* Primary 17B37.
The author was supported in part by FCAR Grant.

In [Lus90] Lusztig has defined a $\mathbf{Q}(v)$-algebra automorphism $\tilde{T}_i : \mathbf{U} \to \mathbf{U}$, which gives a braid group action on $\mathbf{U}$, and used it to give bases of PBW type of $\mathbf{U}^+$. We have

$$\begin{array}{rll} \tilde{T}_i(E_j) = E_j & \text{if} & |i-j| > 1 \\ \tilde{T}_i(E_j) = (E_j E_i - v^{-1} E_i E_j) & \text{if} & |i-j| = 1 \end{array}$$

and $\tilde{T}_i \tilde{T}_j \tilde{T}_i = \tilde{T}_j \tilde{T}_i \tilde{T}_j$ if $|i-j| = 1$ and $\tilde{T}_i \tilde{T}_j = \tilde{T}_j \tilde{T}_i$ if $|i-j| > 1$. Moreover $\tilde{T}_i(E_j) = \tilde{T}_j^{-1}(E_i)$ whenever $|i-j| = 1$.

Let $\mathcal{Q}$ be a quiver of type A_n. Fix a reduced expression $\mathbf{i} = (i_1, i_2, \ldots, i_\nu)$ of w_0, which is adapted to the quiver $\mathcal{Q}$. That is, i_1 is a sink of $\mathcal{Q}$ and for $k > 1$, i_k is a sink of the quiver $s_{i_{k-1}} s_{i_{k-2}} \ldots s_{i_1}(\mathcal{Q})$, where $s_j(\mathcal{Q})$ is the quiver obtained from $\mathcal{Q}$ by reversing the orientation of all the arrows ending at j. The choice of $\mathbf{i}$ gives rise to a total ordering of positive roots $\alpha^1, \alpha^2, \ldots, \alpha^\nu$ where $\alpha^k = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k})$ for $k = 1, \ldots, \nu$. Define

$$E_{\mathbf{i}}^{\mathbf{c}} = E_{i_1}^{(c_1)} \tilde{T}_{i_1} \left(E_{i_2}^{(c_2)} \right) \tilde{T}_{i_1} \tilde{T}_{i_2} \left(E_{i_3}^{(c_3)} \right) \ldots \tilde{T}_{i_1} \tilde{T}_{i_2} \cdots \tilde{T}_{i_{(\nu-1)}} \left(E_{i_\nu}^{(c_\nu)} \right).$$

For $r \in \{1, 2, \ldots, \nu\}$, let $\mathbf{b}(r) \in \mathbf{N}^\nu$ be the vector whose r-th coordinate is 1 and all other coordinates are 0. Then $E_{\mathbf{i}}^{\mathbf{c}} = \prod_{r=1}^{\nu} \left(E_{\mathbf{i}}^{c_r \, \mathbf{b}(r)} \right) = \prod_{r=1}^{\nu} \frac{1}{[c_r]!} \left(E_{\mathbf{i}}^{\mathbf{b}(r)} \right)^{c_r}$.

Theorem 1.1 [Lus90] *The set $B_{\mathbf{i}} = \{ E_{\mathbf{i}}^{\mathbf{c}} \mid \mathbf{c} \in \mathbf{N}^\nu \}$ is a $\mathbf{Q}(v)$-basis of $\mathbf{U}^+$.*

$B_{\mathbf{i}}$ is said to be a basis of PBW type. The elements of $B_{\mathbf{i}}$ of the form $E_{\mathbf{i}}^{\mathbf{b}(r)}$ are called root vectors.

Example 1.2 Let $\mathcal{Q}$ be the quiver $1 \to 2 \leftarrow 3$, and $\mathbf{i} = (2, 1, 3, 2, 1, 3)$. Note that $\mathbf{i}$ is adapted to $\mathcal{Q}$. Then the positive roots with corresponding root vectors are as follows:

$$\begin{array}{lll lll} \alpha^1 & = & \alpha_2 & E_{\mathbf{i}}^{\mathbf{b}(1)} & = & E_2 \\ \alpha^2 & = & \alpha_1 + \alpha_2 & E_{\mathbf{i}}^{\mathbf{b}(2)} & = & E_1 E_2 - v^{-1} E_2 E_1 \\ \alpha^3 & = & \alpha_2 + \alpha_3 & E_{\mathbf{i}}^{\mathbf{b}(3)} & = & E_3 E_2 - v^{-1} E_2 E_3 \\ \alpha^4 & = & \alpha_1 + \alpha_2 + \alpha_3 & E_{\mathbf{i}}^{\mathbf{b}(4)} & = & E_1 E_3 E_2 - v^{-1} E_1 E_2 E_3 \\ & & & & & -v^{-1} E_3 E_2 E_1 + v^{-2} E_2 E_1 E_3 \\ \alpha^5 & = & \alpha_3 & E_{\mathbf{i}}^{\mathbf{b}(5)} & = & E_3 \\ \alpha^6 & = & \alpha_1 & E_{\mathbf{i}}^{\mathbf{b}(6)} & = & E_1. \end{array}$$

Let $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbf{N}^n$ and $E_{\mathbf{d}} = \bigoplus_{i \to j \in \mathcal{Q}^1} \mathrm{Hom}\,(F^{d_i}, F^{d_j})$. $E_{\mathbf{d}}$ is the space of representations of the quiver $\mathcal{Q}$ of dimension $\mathbf{d}$. By Gabriel's theorem, the indecomposable representations of $\mathcal{Q}$ are in 1:1 correspondence with the positive roots. Denote by $\mathbf{e}_\alpha$ the indecomposable representation corresponding to the root α. Then any element of $E_{\mathbf{d}}$ is isomorphic to $\bigoplus_{r=1}^{\nu} c_r \, \mathbf{e}_{\alpha^r}$ for some $\mathbf{c} = (c_1, \ldots, c_\nu) \in \mathbf{N}^\nu$. Since we have a fixed order on positive roots, we may denote this representation by $\mathbf{e}(\mathbf{c})$.

2 α-partitions

In this section we *describe* the notion of α-partitions, for exact definitions see [BS]. Recall that in the Auslander-Reiten quiver $\Gamma_{\mathcal{Q}}$ associated to the quiver $\mathcal{Q}$, the vertices are the indecomposable representations of $\mathcal{Q}$. Thus each vertex of $\Gamma_{\mathcal{Q}}$ corresponds to a unique positive root, by Gabriel's theorem. Let τ be the Auslander-Reiten translation.

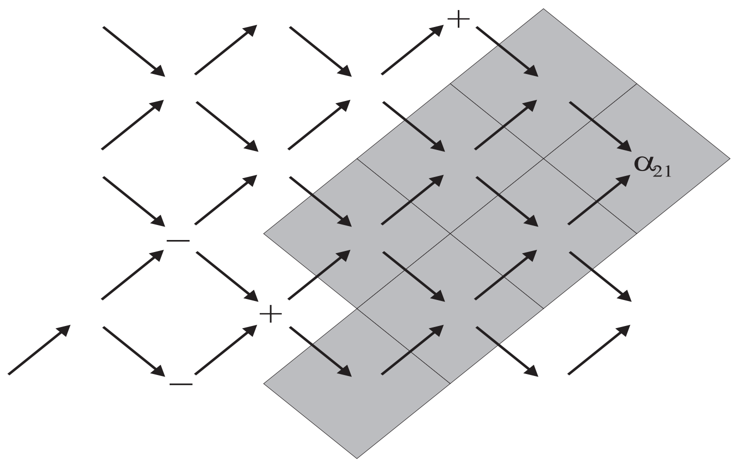
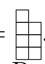


Figure 1 The Auslander-Reiten quiver of $1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5 \rightarrow 6$ with an α^{21}-partition (shaded)

Let λ be a partition and consider its Young diagram, e.g. $\lambda =$ . Place the lower left box of λ at the vertex corresponding to α in the Auslander-Reiten quiver, and perform a rotation of $3\pi/4$ around α. Figure 1 shows an example of type A_6. Then λ is called α-partition if each box of λ corresponds to a non-projective indecomposable representation. Recall that in $\Gamma_{\mathcal{Q}}$ the projective representations are the n vertices that do not have a translate to the left.

The weight $\boldsymbol{\pi}^\lambda = (\pi_1^\lambda, \ldots, \pi_\nu^\lambda)$ of λ is the following element of $\{-1, 0, +1\}^\nu$. Let $s \in \{1, \ldots, \nu\}$. If s is such that one obtains a new partition by adding a box to λ at the position of α^s in $\Gamma_{\mathcal{Q}}$, then let $\pi_s^\lambda = +1$. In figure 1, these positions are marked by a $+$. If $\tau^{-1}(\alpha^s)$ lies in λ and there is no arrow in $\Gamma_{\mathcal{Q}}$ starting at α^s and ending at a vertex lying in λ, then let $\pi_s^\lambda = -1$. In figure 1, these positions are marked by a $-$. Otherwise, let $\pi_s^\lambda = 0$. Note that the positions where $\pi_s^\lambda \neq 0$ lie on a zigzag in the Auslander-Reiten quiver and the $+1$'s correspond to the right corners of the zigzag and the -1's correspond to the left corners.

3 Multiplication by a Root Vector

The following theorem is proved in [BS]. We will not prove it here, we do not even give a precise statement of it, but we will explain it visually in an example. For exact definitions and details, the reader is referred to the original paper.

Theorem 3.1 *For all* $r \in \{1, \ldots, \nu\}$, $\mathbf{c} \in \mathbf{N}^\nu$

$$E_{\mathbf{i}}^{\mathbf{b}(r)}\, E_{\mathbf{i}}^{\mathbf{c}} \;=\; \sum_{\lambda \in \Lambda(r,\mathbf{c})} P(\lambda, \mathbf{c})\, E_{\mathbf{i}}^{\mathbf{c}+\boldsymbol{\pi}^\lambda}$$

with

$$P(\lambda, \mathbf{c}) = \frac{v^{\varphi(\mathbf{c}, \boldsymbol{\pi}^\lambda)}}{(1 - v^{-2})} \prod_{t \in \boldsymbol{\pi}_+^\lambda} (1 - v^{-2(c_t+1)})$$

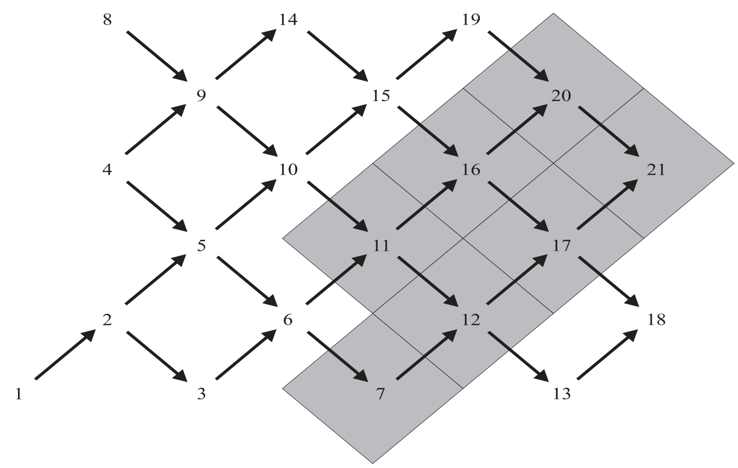


Figure 2 Computation of $P(\lambda, \mathbf{c})$. The ordering of positive roots corresponds to the reduced expression $\mathbf{i} = (1,2,1,4,3,2,1,6,5,4,3,2,1,6,5,4,3,2,6,5,4)$

$$\begin{array}{rcl} \Lambda(r,\mathbf{c}) & = & \text{set of } \alpha^r\text{-partitions } \lambda \text{ such that } \mathbf{c}+\boldsymbol{\pi}^\lambda \in \mathbf{N}^\nu \\ \boldsymbol{\pi}^\lambda & = & (\pi_1^\lambda,\ldots,\pi_\nu^\lambda) \in \{-1,0,1\}^\nu \\ \boldsymbol{\pi}_+^\lambda & = & \{t \mid \pi_t^\lambda = 1\} \\ \varphi(\mathbf{c},\boldsymbol{\pi}^\lambda) & = & \dim \operatorname{Hom}(\mathbf{e}(\mathbf{c}),\mathbf{e}(\boldsymbol{\pi}^\lambda)) - \dim \operatorname{Ext}(\mathbf{e}(\boldsymbol{\pi}^\lambda),\mathbf{e}(\mathbf{c})) \end{array}$$

Where we use the notation $\dim \operatorname{Hom}(\mathbf{e}(\mathbf{c}),\mathbf{e}(\boldsymbol{\pi}^\lambda)) = \sum_t \pi_t^\lambda \dim \operatorname{Hom}(\mathbf{e}(\mathbf{c}),\mathbf{e}(\mathbf{b}(t)))$ *and* $\dim \operatorname{Ext}(\mathbf{e}(\mathbf{c}),\mathbf{e}(\boldsymbol{\pi}^\lambda)) = \sum_t \pi_t^\lambda \dim \operatorname{Ext}(\mathbf{e}(\mathbf{c}),\mathbf{e}(\mathbf{b}(t)))$

We illustrate this theorem in figure 2, in the situation of the example already seen in figure 1. Let $r = 21$. We label the vertices α^s, $s = 1,\ldots,\nu$, of the Auslander-Reiten quiver simply by s. In this example an α^{21}-partition has to be of a shape that fits into a 3×4 rectangle, the rectangle $S(r)$ having corners $5, 7, 19$ and 21. $S(r)$ is the set of all vertices s in Γ_Q such that $\dim \operatorname{Hom}(\mathbf{e}_{\alpha^s},\mathbf{e}_{\alpha^r}) = 1$. Note that the shape of $S(r)$ is not always a complete rectangle, e.g. in figure 2, $S(5) = \{1,2,4,5\}$. Vertex number 5 is the only vertex in $S(21)$ corresponding to a projective module, thus the number of α^{21}-partitions is $\binom{3+4}{3} - 1 = 34$ (see table 1).

Let us calculate $P(\lambda,\mathbf{c})$ for the partition chosen in figure 2. $\varphi(\mathbf{c},\boldsymbol{\pi}^\lambda)$ has a nice description in the figure: $\dim \operatorname{Hom}(\mathbf{e}(\mathbf{c}),\mathbf{e}(\boldsymbol{\pi}^\lambda))$ is the sum of all c_t such that t lies in $S(r)$ but not inside the partition λ; and $\dim \operatorname{Ext}(\mathbf{e}(\boldsymbol{\pi}^\lambda),\mathbf{e}(\mathbf{c}))$ is the sum of all the "translates" $c_{\tau(t)}$ of these c_t. That is

$$\varphi(\mathbf{c},\boldsymbol{\pi}^\lambda) = \sum_{t\in S(r)\setminus\lambda} c_t - c_{\tau(t)}$$

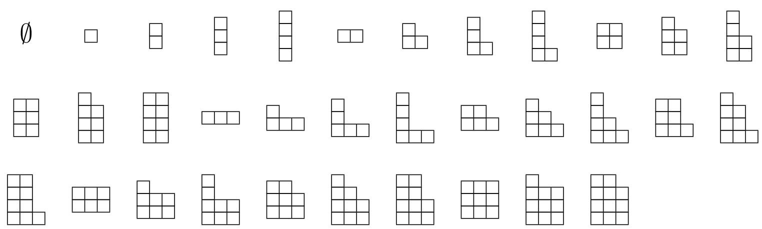
and in our example

$$\varphi(\mathbf{c},\boldsymbol{\pi}^\lambda) = c_5 + c_6 + c_{10} + c_{15} + c_{19} - c_2 - c_4 - c_9 - c_{14}.$$

We already saw in figure 1 that $\boldsymbol{\pi}_+^\lambda = \{6, 19\}$, thus

$$P(\lambda,\mathbf{c}) = \frac{v^{c_5+c_6+c_{10}+c_{15}+c_{19}}}{v^{c_2+c_4+c_9+c_{14}}}\;\frac{(1-v^{-2(c_6+1)})(1-v^{-2(c_{19}+1)})}{(1-v^{-2})}.$$

Table 1 List of all α^{21}-partitions



Remark 3.2 The first step in the proof of the theorem is the calculation of the commutation relations between root vectors. In this step we use Ringel's isomorphism between the twisted Ringel-Hall algebra of representations of the quiver and the plus part of the quantized enveloping algebra [Rin93],[Gre95]. The second step is then to proof the formula by recursion. This is done by a purely combinatorial argument using the Auslander-Reiten quiver.

We do not know how to generalize this result to other quivers. For quivers of type D_n the first step is known (at least in the Ringel-Hall algebra) [Guh00], but even in this case we have not found a combinatorial way to describe the multiplication by a root vector.

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