Cluster algebras and cluster categories

Lecture notes for the
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Contents

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0 Introduction

Cluster algebras were introduced by Fomin and Zelevinsky [FZ] in 2002. Their original motivation was coming from canonical bases in Lie Theory. Today the theory of cluster algebras is connected to various fields of mathematics, including those indicated in Figure 1.

In this minicourse, we will focus on the four framed topics of Figure 1. The first lecture is a short introduction to cluster algebras, lectures two and three are on the connection between cluster algebras and representation theory, which is given by cluster categories and cluster-tilted algebras. We do point out that there is another way to connect cluster algebras to representation theory, using preprojective algebras, which was introduced by Geiss-Leclerc-Schröer, but we will not discuss this here. Lecture four is devoted to cluster algebras from surfaces, especially to the proof of the positivity conjecture for these algebras.

1 Lecture 1: Cluster algebras

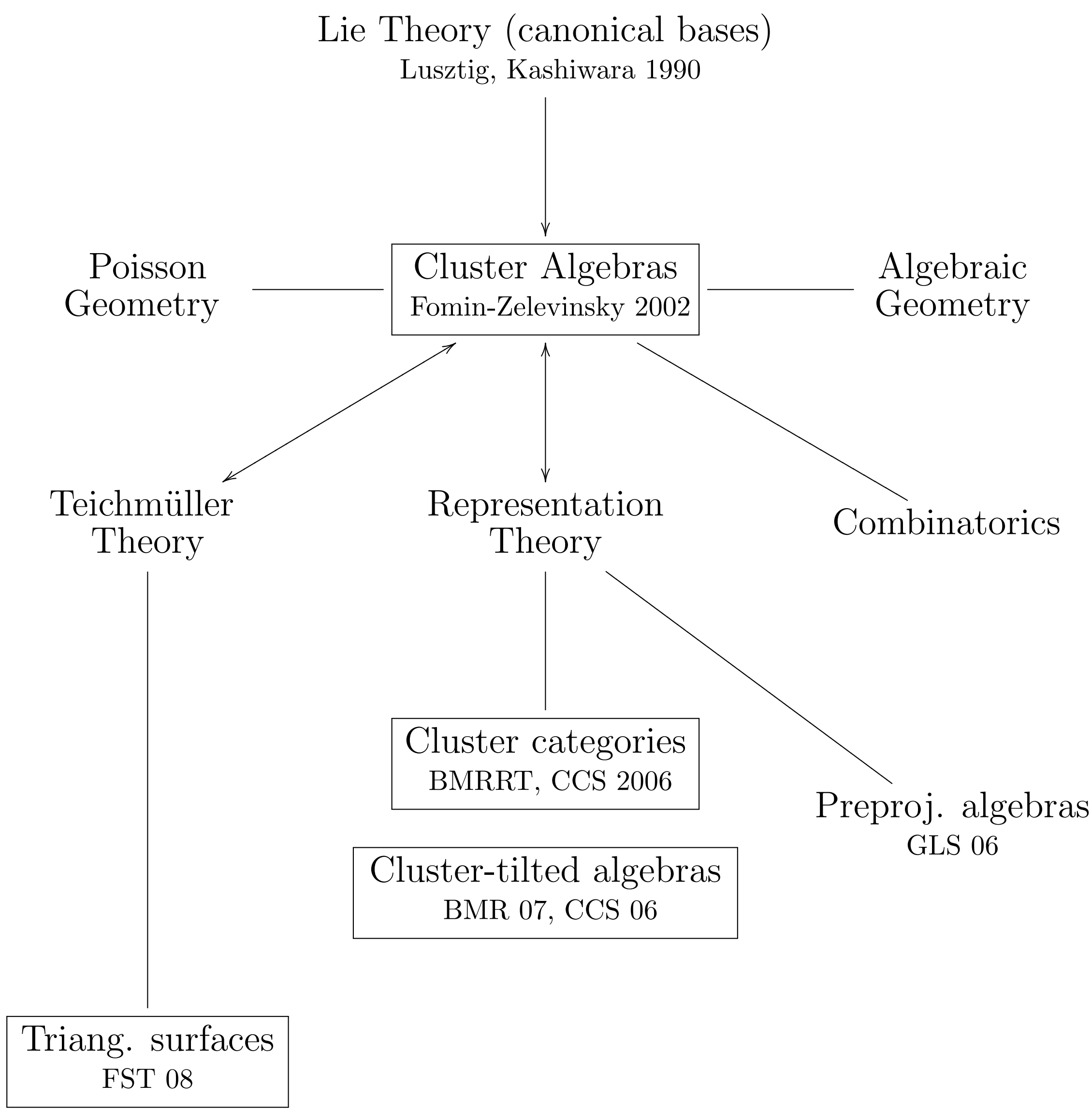
1.1 Definition

The definition of cluster algebras is elementary, but quite complicated. We describe it in this first section.

Cluster algebras $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ are defined by generators and relations, and the set of generators is constructed recursively from some initial data $(\mathbf{x}, \mathbf{y}, Q)$ called *seed*.

For the construction it will be convenient to consider an (infinite) n-regular graph, where n is a fixed positive integer. Recall that a graph is n-regular if each vertex has precisely n neighbors. An example is shown in Figure 2. Each vertex in this graph will correspond to a *seed* $(\mathbf{x}, \mathbf{y}, Q)$ where

- $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a set called *cluster*, and its elements x_i are *cluster variables*



BMRRT: Buan-Marsh-Reineke-Reiten-Todorov
BMR: Buan-Marsh-Reiten
CCS: Caldero-Chapoton-Schiffler
FST: Fomin-Shapiro-Thurston
GLS: Geiss-Leclerc-Schröer

Figure 1: Plan of the lectures

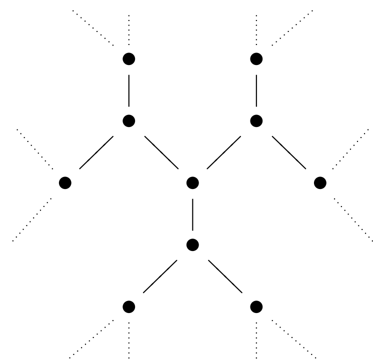


Figure 2: A 3-regular graph

- $\mathbf{y} = (y_1, \ldots, y_n)$ is called *coefficient n-uple* and
- Q is a quiver without loops and 2-cycles.

Each edge of the n-regular graph will correspond to an (involutive) operation called *mutation*.

More precisely, let $\mathbb{P}$ be a free abelian group on generators $u_1, u_2, \ldots, u_\ell$, and let $\mathbb{ZP}$ be its group ring, that is, $\mathbb{ZP}$ is the ring of Laurent polynomials in $u_1, u_2, \ldots, u_\ell$. $\mathbb{ZP}$ will be the ground ring for the cluster algebra. Let $\mathcal{F} = \mathbb{QP}(x_1, \ldots, x_n)$ be a field of rational functions in n-variables over $\mathbb{QP}$. This field is called the *ambient field* of the cluster algebra $\mathcal{A}$. Thus $\mathcal{A}$ is a $\mathbb{ZP}$-subalgebra of $\mathcal{F}$.

Initial seed $(\mathbf{x}, \mathbf{y}, Q)$

$\mathbf{x} \;=\; \{x_1, \ldots, x_n\}$ transcendence basis for $\mathcal{F}$

$\mathbf{y} \;=\; (y_1, \ldots, y_n)$ $y_i \in \mathbb{ZP}$

Q a quiver with n vertices and without loops $\circ\circlearrowright$ and 2-cycles $\circ \rightleftarrows \circ$

The cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ is completely determined by one (arbitrary) seed. The set of generators is constructed recursively from the initial seed using mutations. The generators will be called *cluster variables* and the n cluster variables in the initial seed are called *initial* cluster variables.

Mutations μ_k, $k = 1, 2, \ldots, n$ A mutation μ_k transforms a given seed $(\mathbf{x}, \mathbf{y}, Q)$ into a new seed $(\mathbf{x}', \mathbf{y}', Q')$ where

- $\mathbf{x}'$ is obtained from $\mathbf{x}$ by replacing one variable by a new one, $\mathbf{x}' = \mathbf{x} \setminus \{x_k\} \cup \{x'_k\}$, and x'_k is defined by the following *exchange relation*

$$x_k x'_k = y_+ \prod_{\alpha: i \to k} x_i \; + \; y_- \prod_{\alpha: i \leftarrow k} x_i \tag{1}$$

  where y_+, y_- are some coefficients, that we do not specify here,

- $\mathbf{y}' = (y'_1, \ldots, y'_n)$ is a new coefficient n-uple that we also do not specify here. Let us just point out that $\mathbf{y}'$ depends only on $\mathbf{y}$ and Q,

- The quiver Q' is obtained from Q in three steps:

  1. for every path $i \to k \to j$ add one arrow $i \to j$,
  2. reverse all arrows at k,
  3. delete 2-cycles.

See Figure 3 for three examples.

Mutations are involutions, that is, $\mu_k \mu_k(\mathbf{x}, \mathbf{y}, Q) = (\mathbf{x}, \mathbf{y}, Q)$.

Note that Q' only depends on Q, that $\mathbf{y}'$ depends on $\mathbf{y}$ and Q, and that $\mathbf{x}'$ depends on the whole seed $(\mathbf{x}, \mathbf{y}, Q)$.

Let us return to the picture of the n-regular graph. The initial seed is one of the vertices in this graph. Applying the first n mutations to this seed, we get the n neighbors of this vertex in the graph, each of which contains exactly one new cluster variable. So at this stage we have $2n$ cluster variables. Now we can continue mutating these new seeds, and at every step we construct a "new" cluster variable. It may happen, that we obtain a seed that has already appeared previously in this process. In that case we identify the two corresponding vertices in the n-regular graph, and the actual exchange graph is a quotient of the graph in Figure 2. Such a repetition may happen but it does not have to, and in general the number of seeds is infinite. The whole pattern is determined by the initial seed.

The *cluster algebra* $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by the set of all cluster variables obtained by sequences of mutations from the initial seed $(\mathbf{x}, \mathbf{y}, Q)$.

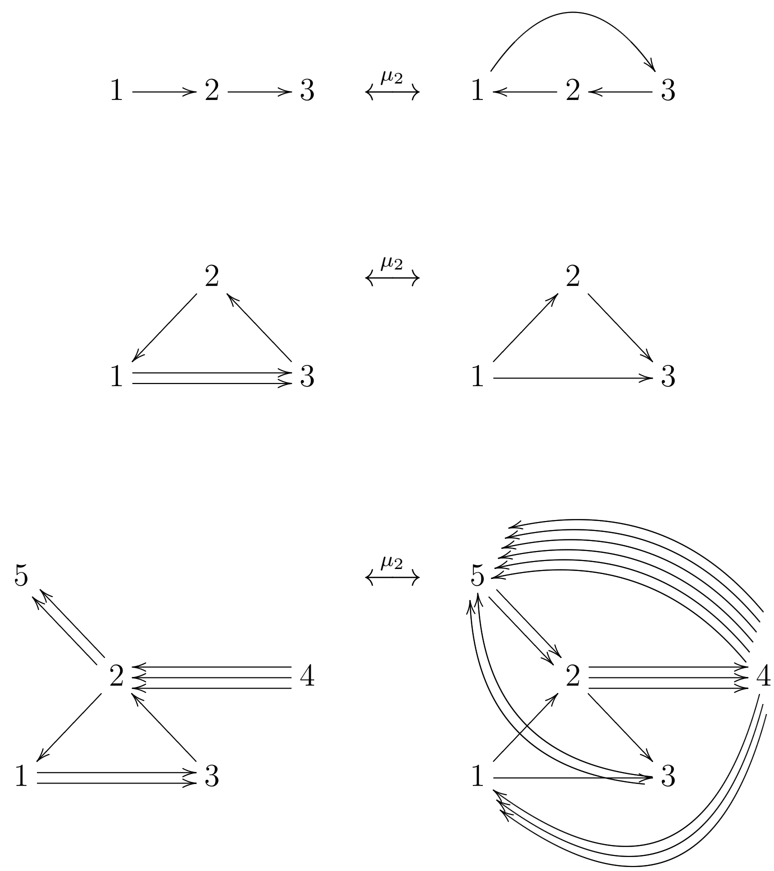


Figure 3: Examples of quiver mutations

Remark 1.1 Fomin and Zelevinsky define cluster algebras in a more general way using skew-symmetrizable matrices instead of quivers. The quiver definition corresponds to the special case where the matrices are skew-symmetric.

Example 1 $(\mathbf{x}, \mathbf{y}, Q) = (\{x_1, x_2\}, \{1, 1\}, 1 \to 2)$

Since the coefficients in this example are trivial, the coefficients in any seed will be $\{1, 1\}$. We therefore omit them in the computation below.

$$\{x_1, x_2\}, 1 \to 2$$

Apply mutation μ_1 (note that the empty product is 1, by definition)

$$\left\{\frac{x_2 + 1}{x_1}, x_2\right\}, 1 \leftarrow 2$$

Apply mutation μ_2

$$\left\{\frac{x_2 + 1}{x_1}, \frac{x_2 + 1 + x_1}{x_1 x_2}\right\}, 1 \to 2$$

Apply mutation μ_1. Let us do this step in detail. We get

$$\frac{\frac{x_2+1+x_1}{x_1 x_2} + 1}{\frac{x_2+1}{x_1}} = \frac{(x_2 + 1 + x_1 + x_1 x_2)x_1}{x_1 x_2 (x_2 + 1)} = \frac{(x_2 + 1)(x_1 + 1)}{x_2 (x_2 + 1)}$$

Thus the new seed is

$$\left\{\frac{x_1 + 1}{x_2}, \frac{x_2 + 1 + x_1}{x_1 x_2}\right\}, 1 \leftarrow 2$$

Apply mutation μ_2

$$\left\{\frac{x_1 + 1}{x_2}, x_1\right\}, 1 \to 2$$

Apply mutation μ_1

$$\{x_2, x_1\}, 1 \leftarrow 2$$

Continuing the process from here will not yield new cluster variables. Thus in this case, there are exactly 5 cluster variables

$$x_1, x_2, \frac{x_2 + 1}{x_1}, \frac{x_2 + 1 + x_1}{x_1 x_2}, \frac{x_1 + 1}{x_2}.$$

1.2 Laurent Phenomenon and positivity conjecture

Theorem 1.2 *[FZ] Let x be any cluster variable in $\mathcal{A}$. Then x can be written as a Laurent polynomial in any cluster $\mathbf{x} = \{x_1, \ldots, x_n\}$, that is,*

$$x = f(x_1, \ldots, x_n)/(x_1^{d_1} \cdots x_n^{d_n})$$

with $d_i \geq 0$ and $f \in \mathbb{ZP}[x_1, \ldots, x_n]$ is a polynomial.

Remark 1.3 This is a surprising result, since, a priori, the cluster variables are rational functions in the variables $x_1, \ldots, x_n$. The theorem says that the denominators of these rational functions are actually monomials. This means that at each mutation, when we have to divide a binomial of cluster variables by a certain cluster variable x', the numerator of that cluster variable x' is actually a factor of that binomial. Note that the numerator of x' may be a complicated polynomial. We have already observed this phenomenon in the third step of example 1.

Conjecture 1.4 *[FZ] The coefficients of the polynomial f in the Laurent phenomenon are non-negative integer linear combinations of elements in $\mathbb{P}$, that is,*

$$f \in \mathbb{Z}_{\geq 0}\mathbb{P}[x_1, \ldots, x_n].$$

Remark 1.5 In Lecture 4, we will see that this conjecture holds in the case of cluster algebras from surfaces.

1.3 Classifications

Especially nice cluster algebras are those of finite type, of finite mutation type and of acyclic type cluster algebras. We explain here how these types overlap.

Definition 1 *(a) A cluster algebra is of* finite type *if the number of clusters is finite.*

(b) A cluster algebra is of finite mutation type *if the number of quivers that appear in the seeds is finite.*

(c) A cluster algebra is acyclic *if there exists a seed $(\mathbf{x}, \mathbf{y}, Q)$ such that the quiver is without oriented cycles.*

Fomin-Zelevinsky showed that the finite-type cluster algebras are classified by the Dynkin diagrams. Since we are considering only cluster algebras given by quivers, we only get the simply laced Dynkin diagrams.

Theorem 1.6 *[FZ2] A cluster algebra is of finite type if and only if there exists a seed* $(\mathbf{x}, \mathbf{y}, Q)$ *with* Q *a quiver of Dynkin type* $\mathbb{A}, \mathbb{D}$ *or* $\mathbb{E}$*..*

Theorem 1.7 *[FeShTu] A cluster algebra* $\mathcal{A}$ *is of finite mutation type if and only if*

- $\mathcal{A}$ *is coming from a surface (see Lecture 4), or*
- $n \leq 2$*, or*
- $\mathcal{A}$ *is one of 11 exceptional types.*

The overlaps of the vaious classes of cluster algebras for $n \geq 3$ are illustrated in Figure 4. Acylic and mutation finite types have in common the finite types $\mathbb{A}, \mathbb{D}, \mathbb{E}$ and the tame types corresponding to the extended Dynkin diagrams $\widetilde{\mathbb{A}}, \widetilde{\mathbb{D}}, \widetilde{\mathbb{E}}$. Other acyclic types are called wild. The 11 exceptions in Theorem 1.7 are indicated by dots; 9 of them correspond to root systems of certain E-types.

Different types of cluster algebras

$\tilde{A}$ $\tilde{D}$

A D

$\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$

X_6, X_7 $E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$ E_6, E_7, E_8

Figure 4: Different types of cluster algebras

2 Lecture 2: Cluster categories

If $\mathcal{A}$ is an acyclic cluster algebra and $(\mathbf{x}, \mathbf{y}, Q)$ is a seed whose quiver Q has no oriented cycles, then we can associate to $\mathcal{A}$ a triangulated category, the *cluster category*, which is defined in terms of the quiver Q.

2.1 Definition

Let k be field, which we suppose to be algebraically closed for simplicity. Let $A = kQ$ be the path algebra of the quiver Q. This algebra has a basis given by the set of paths in the quiver, and its multiplication is given on that basis by concatenation of paths, where one uses the convention that if two paths can't be concatenated then their product is zero.

For example, if Q is the quiver $1 \xleftarrow{\beta} 2 \xleftarrow{\alpha} 3$ then the basis of A is equal to $\{e_1, e_2, e_3, \alpha, \beta, \alpha\beta\}$ where e_i denotes the constant path at the vertex i. Examples for the multiplication are

$$e_2\alpha = 0, \quad e_3\alpha = \alpha, \quad \beta\alpha = \beta\alpha.$$

In this example, the path algebra A is isomorphic to the algebra of lower triangular matrices with entries in k; the isomorphism is given by sending the scalars of a path from i to j to the entry at the position ij in the matrix.

Let $\mathcal{D} = \mathcal{D}^b(\operatorname{mod} A)$ be the derived category of bounded complexes of modules of A. Derived categories are very complicated categories in general, but since A is a hereditary algebra the derived category has a very simple structure: The indecomposable objects of $\mathcal{D}$ are of the form $M[i]$ where M is an indecomposable A-module and $i \in \mathbb{Z}$ is the shift in $\mathcal{D}$.

The morphisms are given by

$$\operatorname{Hom}_{\mathcal{D}}(M[i], N[j]) = \operatorname{Ext}_A^{j-i}(M, N) = \begin{cases} \operatorname{Hom}_A(M, N) & \text{if } j = i \\ \operatorname{Ext}_A^1(M, N) & \text{if } j = i + 1 \\ 0 & \text{otherwise,} \end{cases}$$

where the last equation holds since A is hereditary.

Remark 2.1 $\mathcal{D}$ has Serre duality, which means that there is an autoequivalence τ called the Auslander-Reiten translation such that

$$\operatorname{Hom}_{\mathcal{D}}(M, N[1]) \cong \operatorname{D} \operatorname{Hom}_{\mathcal{D}}(N, \tau M),$$

where $\operatorname{D} = \operatorname{Hom}_k(-, k)$ is the standard duality.

Definition 2 *[BMRRT] Let $F = \tau_{\mathcal{D}}^{-1}[1]$. The cluster category $\mathcal{C} = \mathcal{C}_A$ is the orbit category $\mathcal{D}/F$ of the functor F in $\mathcal{D}$. Its (isoclasses of) objects are the F-orbits $\widetilde{M} = (F^i M)_{i \in \mathbb{Z}}$ of objects $M \in \mathcal{D}$, and its morphisms are by definition*

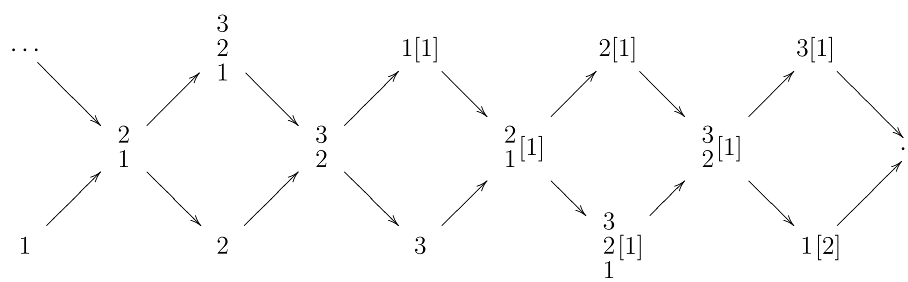
$$\mathrm{Hom}_{\mathcal{C}}(\widetilde{M}, \widetilde{N}) = \oplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(M, F^i N)$$

Remark 2.2 $\mathcal{C}$ is a triangulated category [K] with Auslander-Reiten triangles. The shift and the Auslander-Reiten translation in $\mathcal{C}$ are induced by shift and Auslander-Reiten translation in $\mathcal{D}$. Thus

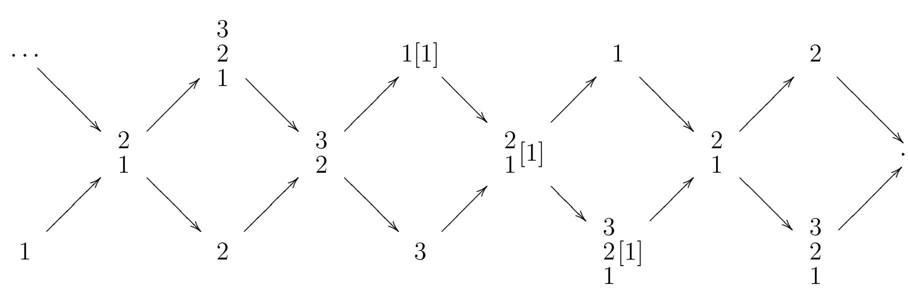
$$\tau_{\mathcal{C}} \widetilde{M} = \widetilde{\tau_{\mathcal{D}} M} \qquad \text{and} \qquad \widetilde{M[1]_{\mathcal{D}}} = \widetilde{M}[1]_{\mathcal{C}}$$

We give an explicit example of the Auslander-Reiten quivers of $\mathcal{D}$ and $\mathcal{C}$. The vertices in these quivers correspond to the (isoclasses of) indecomposable objects in the categories. The Auslander-Reiten translation τ^{-1} is the horizontal translation to the right.

Example 2.3 *Let $Q : 1 \leftarrow 2 \leftarrow 3$. Then the derived category is*



To construct the cluster category, we must identify objects that lie in the same F-orbit. For example, the object 1 lies in the same orbit as $F1 = \tau^{-1}1[1] = 2[1]$, the object $\begin{smallmatrix}2\\1\end{smallmatrix}$ lies in the same orbit as $\tau^{-1} \begin{smallmatrix}2\\1\end{smallmatrix}[1] = \begin{smallmatrix}3\\2\end{smallmatrix}[1]$. Thus the cluster category is



where one has to identify the objects that have the same label. Hence the Auslander-Reiten quiver in this example lies on a Moebius strip.

Remark 2.4 *Let us make the following simple observations.*

1. *The indecomposable objects in $\mathcal{C}$ are the indecomposable objects in* $\mathrm{mod}\ A$ *plus n extra objects which are the shifts of the n projective indecomposable modules. Thus the cluster category is just a little larger (n indecomposables more) than the module category.*

2. $\tau_{\mathcal{C}} = [1]_{\mathcal{C}}$.
   Proof: $\tau_{\mathcal{C}}\widetilde{M} = \widetilde{\tau_{\mathcal{D}}M} = \widetilde{\tau_{\mathcal{D}}FM} = \widetilde{M[1]_{\mathcal{D}}} = \widetilde{M}[1]_{\mathcal{C}}$.

3. *$\mathcal{C}$ has Serre duality:* $\mathrm{Ext}^1_{\mathcal{C}}(\widetilde{M}, \widetilde{N}) \cong \mathrm{D}\,\mathrm{Hom}_{\mathcal{C}}(\widetilde{N}, \tau\widetilde{M})$.

   Proof: $\mathrm{Ext}^1_{\mathcal{C}}(\widetilde{M}, \widetilde{N}) \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(\widetilde{M}, \widetilde{N}[1]) = \oplus\,\mathrm{Hom}_{\mathcal{D}}(M, F^iN[1])$ and $\mathrm{D}\,\mathrm{Hom}_{\mathcal{C}}(\widetilde{N}, \tau\widetilde{M}) = \oplus\mathrm{D}\,\mathrm{Hom}_{\mathcal{D}}(F^iN, \tau M)$.
   The result now follows from Serre duality in $\mathcal{D}$.

4. *$\mathcal{C}$ is 2-Calabi Yau:* $\mathrm{Ext}^1_{\mathcal{C}}(\widetilde{M}, \widetilde{N}) = \mathrm{D}\,\mathrm{Ext}^1_{\mathcal{C}}(\widetilde{N}, \widetilde{M})$.
   Proof: $\mathrm{Ext}^1_{\mathcal{C}}(\widetilde{M}, \widetilde{N}) \overset{(3)}{=} \mathrm{D}\,\mathrm{Hom}_{\mathcal{C}}(\widetilde{N}, \tau\widetilde{M}) \overset{(2)}{=} \mathrm{D}\,\mathrm{Hom}_{\mathcal{C}}(\widetilde{N}, \widetilde{M}[1]) = \mathrm{D}\,\mathrm{Ext}^1_{\mathcal{C}}(\widetilde{N}, \widetilde{M})$.

From now on we will use notation without tilde for the objects in $\mathcal{C}$.

2.2 Cluster-tilting theory

Let n be the number of vertices of the quiver Q.

An object $T \in \mathcal{C}$ is called *cluster-tilting* if

1. $\mathrm{Ext}^1_{\mathcal{C}}(T, T) = 0$
2. $T = \oplus_{i=1}^n T_i$, with T_i indecomposable and $T_i \not\cong T_j$.

Remark 2.5 Objects that satisfy condition (1) are called *rigid*.

Condition (2) is a maximality condition, since one can show that if T has more that n non-isomorphic indecomposable summands then T is not rigid.

Theorem 2.6 *[BMRRT] Let T be a cluster-tilting object in $\mathcal{C}$ and let M be an indecomposable summand of T, and $T = \overline{T} \oplus M$. Then there exists a unique indecomposable $M' \ncong M$ such that $\overline{T} \oplus M'$ is cluster-tilting. Moreover, there are unique triangles in $\mathcal{C}$*

$$\begin{array}{llcll} M' & \to \oplus_{i \in I} B_i & \to & M & \to & M'[1] \\ M & \to \oplus_{i \in I'} B'_i & \to & M' & \to & M[1] \end{array}$$

with B_i, B'_i indecomposable summands of T.

Remark 2.7 Such a complement M' does not always exist in the module category of A; one really needs the slightly bigger cluster category for Theorem 2.6.

2.3 Relation to cluster algebras

Theorem 2.8 *[BMRRT, CK] Let Q be an acyclic quiver, $\mathcal{C} = \mathcal{C}_{kQ}$ its cluster category and $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ a cluster algebra associated to Q.*

1. *There are bijections*

$$\begin{array}{ccc} \{\text{indecomposable rigid objects in } \mathcal{C}\} & \longrightarrow & \{\text{cluster variables in } \mathcal{A}\} \\ M & \longmapsto & x_M \end{array}$$

$$\begin{array}{ccc} \{\text{cluster-tilting objects in } \mathcal{C}\} & \longrightarrow & \{\text{clusters in } \mathcal{A}\} \\ T = T_1 \oplus \cdots \oplus T_n & \longmapsto & \mathbf{x}_T = \{x_{T_1}, \ldots, x_{T_n}\} \end{array}$$

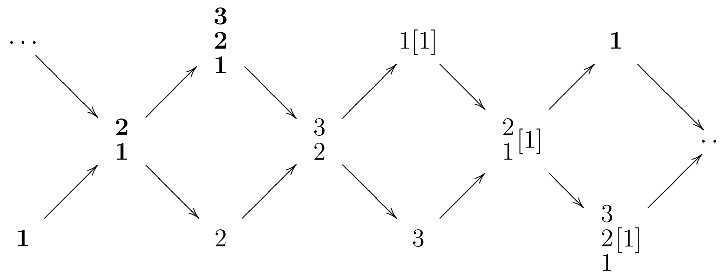
2. *Let $M, N \in \mathcal{C}$ be rigid and indecomposable. Then $\dim \operatorname{Ext}^1_{\mathcal{C}}(M, N) = 1$ if and only if x_M and x_M form an exchange pair, that is, there exists a cluster $\mathbf{x}$ and a mutation μ such that $\mu(\mathbf{x}) = \mathbf{x} \setminus \{x_M\} \cup \{x_N\}$.*

   Moreover, in this situation the exchange relation (1) is given by

$$x_M x_N = * \prod_{i \in I} x_{B_i} + * \prod_{i \in I'} x_{B'_i}$$

   where B_i, B'_i are as in Theorem 2.6 and $*$ are some coefficients.

Example 2.9 *In the cluster category of the quiver $1 \leftarrow 2 \leftarrow 3$, let $T = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$. We have indicated this object in bold type in the figure below.*



If we mutate this object in $\begin{smallmatrix}2\\1\end{smallmatrix}$ *, we get*

$$\mu_{\substack{2\\1}} T = 1 \oplus \begin{matrix}3\\2\\1\end{matrix} \oplus 3$$

with the corresponding exchange triangles

$$\begin{array}{ccccccc} \begin{matrix}2\\1\end{matrix} & \to & \begin{matrix}3\\2\\1\end{matrix} & \to & 3 & \to & \begin{matrix}2\\1\end{matrix}[1] \\ 3 & \to & 1 & \to & \begin{matrix}2\\1\end{matrix} & \to & 3\,[1]. \end{array}$$

Moreover, the exchange relation is

$$x_3\, x_{\substack{2\\1}} = *\, x_{\substack{3\\2\\1}} + *\, x_1 \ .$$

3 Lecture 3: Cluster-tilted algebras

Cluster-tilted algebras are endomorphism algebras of cluster-tilting objects in cluster categories.

More precisely, let A be a path algebra of a quiver without oriented cycles, $\mathcal{C}$ the cluster category and T a cluster-tilting object in $\mathcal{C}$.

Definition 3 *[BMR] The endomorphism algebra* $\mathrm{End}_{\mathcal{C}}\, T$ *is called* cluster-tilted algebra*.*

Recall from Lecture 2 that there is a bijection

$$\{\text{seeds } (\mathbf{x}, \mathbf{y}, Q)\ \} \longrightarrow \{\text{cluster-tilting objects } T\}$$

Thus by the above definition we associate a finite dimensional algebra, the cluster-tilted algebra $\mathrm{End}_{\mathcal{C}}\, T$, to every cluster in the cluster algebra. Moreover, the cluster-tilted algebra can be given as a quotient of a path algebra, thus $\mathrm{End}_{\mathcal{C}}\, T = kQ/I$ and the quiver Q is the same as the quiver in the seed $(\mathbf{x}, \mathbf{y}, Q)$. Even better,

the quiver Q determines the cluster-tilted algebra kQ/I (up to isomorphism).

3.1 Relations in finite type

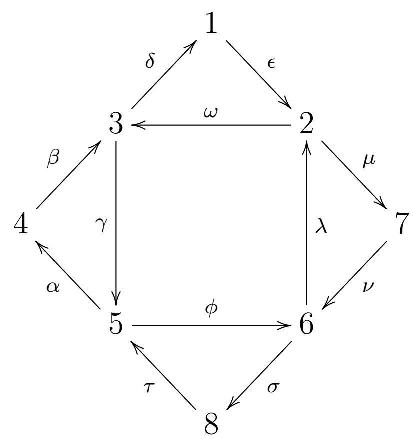
If the cluster algebra is of finite type (i.e. A is of finite representation type) then one can explicitly describe a system of relations for the ideal I using the notion of shortest paths as follows:

By definition, a *shortest path* in the quiver Q is an oriented path (with no repeated arrow) contained in an induced subgraph of Q which is a cycle.

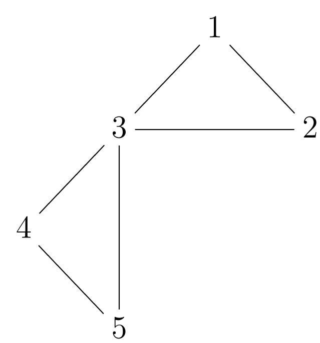
For every arrow $i \stackrel{\alpha}{\to} j$ in Q define a relation ρ_α to be the sum of all shortest paths from j to i.

Proposition 3.1 *[CCS2, BMR2] The ideal I is generated by the set of all ρ_α.*

For example, consider the arrow α in the quiver



then $\rho_\alpha = \beta\gamma$ since all other paths from 4 to 5 are not shortest. For example, the path $\beta\delta\epsilon\omega\gamma$ induces the subgraph



which is not a cycle. On the other hand $\rho_\gamma = \alpha\beta + \phi\lambda\omega$.

The ideal of the cluster-tilted algebra in this example is generated by 12 relations, 8 monomial relations induced by the 8 exterior arrows and 4 binomial relations induced by the 4 interior arrows.

Remark 3.2 *In general, the relations can be described using partial derivatives of potentials [DWZ].*

3.2 Equivalence of categories

The following theorem completely determines the module category of a cluster-tilted algebra provided that the cluster category is known.

Theorem 3.3 *[BMR] The functor* $\operatorname{Hom}_{\mathcal{C}}(T, -)$ *induces an equivalence of categories*

$$\mathcal{C}/\operatorname{add}\tau T \longrightarrow \operatorname{mod}\operatorname{End}_{\mathcal{C}} T$$

Note that, roughly speaking, $\mathcal{C}/\operatorname{add}\tau T$ is obtained from the cluster category by deleting the n indecomposable objects $\tau T_1, \ldots, \tau T_n$.

Corollary 3.4 *Let* $B = \operatorname{End}_{\mathcal{C}_A} T$ *be a cluster-tilted algebra. The following are equivalent*

(a) B is of finite representation type.

(b) A is of finite representation type.

(c) B (and A) is of Dynkin type $\mathbb{A}, \mathbb{D}$ *or* $\mathbb{E}$*.*

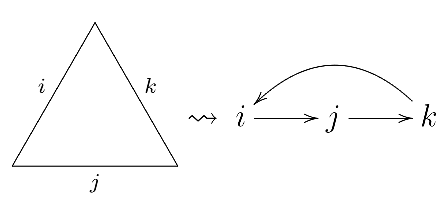
(d) The corresponding cluster algebra is of finite type.

Moreover, in this case, A and B have the same number of (isoclasses of) indecomposable modules, and this number is equal to the number of positive roots, hence $n(n+1)/2$ *in type* $\mathbb{A}_n$*,* $(n-1)n$ *in type* $\mathbb{D}_n$ *and* $36, 63, 120$ *in types* $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$*, respectively.*

3.3 Cluster-tilted algebras of type $\mathbb{A}_n$

We briefly discuss here an alternative definition of cluster-tilted algebras (and cluster categories) of type $\mathbb{A}_n$, using diagonals of a regular polygon. This definition was given in [CCS] independently of the papers [BMRRT, BMR].

We associate a quiver Q to a triangulation $T = \{1, 2, \ldots, n\}$ of a polygon with $n+3$ vertices as follows. The vertices of Q are the diagonals $\{1, 2, \ldots, n\}$ of T and there is an arrow $i \to j$ in Q_1 precisely if the diagonals i and j bound a triangle in which j lies counterclockwise of i.

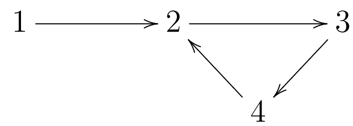


Theorem 3.5 *[CCS] The cluster-tilted algebras of type $\mathbb{A}_n$ are precisely those obtained from the triangulations of a polygon with $n+3$ vertices.*

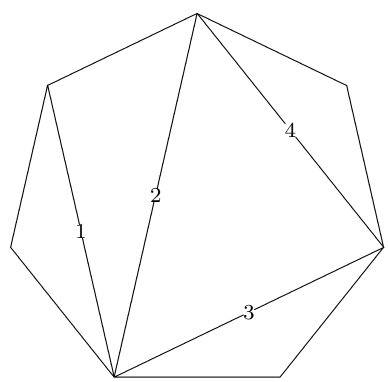
Moreover the module category is completely determined by the triangulation, where we have the following dictionary

diagonals τ not in T	$\leftrightarrow$	indecomposable modules M_τ
crossing between τ and T	$\leftrightarrow$	dimension vector of M_τ
elementary clockwise rotation	$\leftrightarrow$	Auslander-Reiten translation
intersection numbers	$\leftrightarrow$	$\dim \operatorname{Ext}^1_{\mathcal{C}}$ in the cluster category.

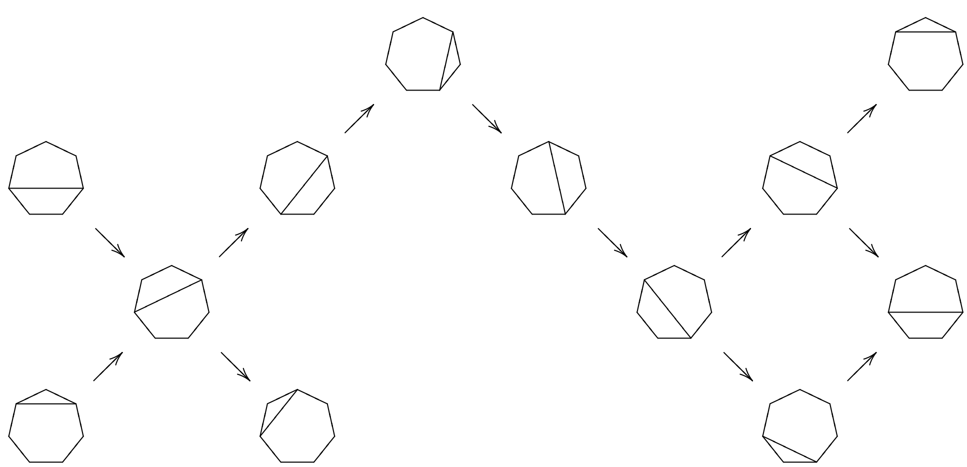
Example 3.6 *Let Q be the quiver*



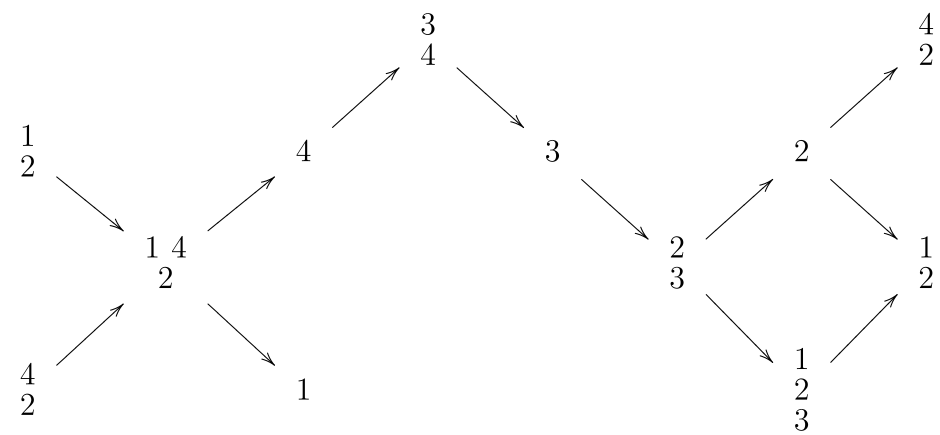
associated to the triangulation T



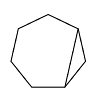
Then the Auslander-Reiten quiver is



which translates into module notation as



where one has to identify the two representations labeled $\begin{smallmatrix}1\\2\end{smallmatrix}$ and the two representations labeled $\begin{smallmatrix}4\\2\end{smallmatrix}$, so that the Auslander-Reiten quiver has the shape of a Moebius strip.

For example, the diagonal  crosses the diagonal 3 and 4 of the triangulation T. Thus the module M_τ has dimension vector (0.0.1.1), hence $M_\tau = \begin{smallmatrix}3\\4\end{smallmatrix}$.

Remark 3.7 *For type $\mathbb{D}_n$ cluster-tilted algebras, there is a similar description in terms of a punctured polygon [S].*

3.4 Relation to tilted algebras

Tilted algebras are endomorphism algebras of tilting modules over a hereditary algebra.

As before, let A be the path algebra of a quiver without oriented cycles, and let T be a tilting A-module, that is,

1. $\mathrm{Ext}^1_A(T, T) = 0$
2. $T = \oplus_{i=1}^n T_i$, with T_i indecomposable and $T_i \not\cong T_j$.

Note the similarity to the definition of a cluster-tilting object in the cluster category. Therefore, it is not surprising that there is a strong connection between tilted algebras and cluster-tilted algebras. To describe this connection, we need one more definition.

Definition 4 *For any algebra C of global dimension at most 2 (e.g. C tilted) we define the* relation extension algebra $\widetilde{C}$ *as the semi-direct product*

$$\widetilde{C} = C \ltimes \operatorname{Ext}^2_C(DC, C)$$

where $D = \operatorname{Hom}_C(-, k)$ denotes the standard duality.

Thus as a vector space $\widetilde{C}$ is the direct sum $C \oplus \operatorname{Ext}^2_C(DC, C)$ and the multiplication is given by

$$(c, e)(c', e') = (cc', ce' + ec').$$

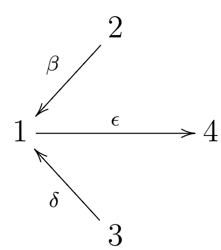
Theorem 3.8 *[ABS1] There is a surjective map*

$$\begin{array}{rcccl} \phi & : & \{\text{tilted algebras}\} & \longrightarrow & \{\text{cluster-tilted algebras}\} \\ & & C & \longmapsto & \widetilde{C} = C \ltimes \operatorname{Ext}^2_C(DC, C) \end{array}$$

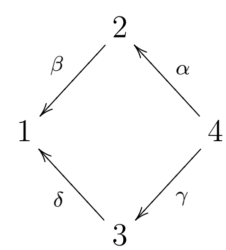
which preserves the type.

Remark 3.9 *This map is not injective. For example, consider the two quivers*

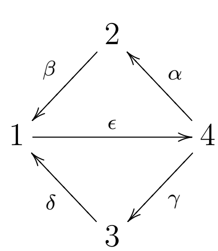
Q_1 :



and Q_2 :



and let C_1 be the algebra given by Q_1 with relations $\beta\epsilon = 0, \delta\epsilon = 0$ and let C_2 be the algebra given by Q_2 with the relation $\alpha\beta = \gamma\delta$. Then C_1 and C_2 are tilted algebras of type $\mathbb{D}_4$ that have the same cluster-tilted algebra $\widetilde{C_1} = \widetilde{C_2}$ which is given by the quiver



and relations

$$\begin{aligned} \beta\epsilon &= 0, \\ \delta\epsilon &= 0, \\ \epsilon\alpha &= 0, \\ \epsilon\gamma &= 0, \\ \alpha\beta &= \gamma\delta \end{aligned}$$

Remark 3.10 *Theorem 3.8 has been generalized to m-iterated tilted algebras and m-cluster-tilted algebras by Fernandez-Pratti-Trepode.*

In [BFPPT] the authors show that if one replaces the tilted algebra by an iterated tilted algebra of global dimension 2, then the relation extension is a quotient of the cluster-tilted algebra, but not always equal to it.

Having establishd Theorem 3.8, it is natural to ask what properties of tilted algebras can be carried over to cluster-tilted algebras. We give a few results in this direction.

3.5 The quiver of a cluster-tilted algebra

Theorem 3.11 *[ABS1] Let C be a tilted algebra and $\widetilde{C}$ its cluster-tilted algebra. Let Q be the quiver of C and $\widetilde{Q}$ the one of $\widetilde{C}$. Then the vertices of $\widetilde{Q}$ are the same as the vertices of Q, and the number of arrows $i \to j$ in $\widetilde{Q}$ equals the number of arrows $i \to j$ in Q plus the number of relations $j \rightsquigarrow \cdots \rightsquigarrow i$ in a minimal system of relations for C.*

3.6 Local slices

Tilted algebras are characterized by the existence of a complete slice in their module category. A complete slice is a certain connected, convex subquiver of the Auslander-Reiten quiver that intersects every τ-orbit precisely once. Here convex means that when $M = M_0 \to M_1 \to \cdots \to M_t = N$ is a sequence of non-zero morphisms between indecomposable modules, with M and N in the complete slice, then every M_i lies in the slice.

The map of Theorem 3.8 induces an embedding of the module categories $\operatorname{mod} C \hookrightarrow \operatorname{mod} \widetilde{C}$. This embedding does not commute with Auslander-Reiten translations in general. However, the modules that are close to a complete slice embed nicely, in particular, the image of a complete slice in $\operatorname{mod} C$ is still a connected subquiver of the Auslander-Reiten quiver $\operatorname{mod} \widetilde{C}$.

The notion of convexity does not make much sense in the module category of the cluster-tilted algebra, since it contains cyclic sequences of non-zero morphisms and indecomposable modules $M = M_0 \to M_1 \to \cdots \to M_t = M$. But the image of a complete slice is still "loccally convex" in a certain sense, which needs naturally to the concept of local slices.

Theorem 3.12 *[ABS2] Let B be a cluster-tilted algebra and ϕ the map of Theorem 3.8.*

(a) *A tilted algebra C lies in the fiber $\phi^{-1}(B)$ if and only if B has a local slice Σ such that C is the quotient of B by the annihilator of Σ.*

(b) *For every tilted algebra C in the fiber $\phi^{-1}(B)$ and every slice Σ of C, the image of Σ under the embedding $\operatorname{mod} C \hookrightarrow \operatorname{mod} B$ is a local slice, and every local slices arises this way.*

In contrast to tilted algebras, where, generally speaking, a complete slice is a *special* configuration inside the Auslander-Reiten quiver, a cluster-tilted algebra has many local slices. In fact, one can show that in finite representation type, *every* indecomposable module lies on a local slice.

Again, in contrast to tilted algebras, cluster-tilted algebras are not characterized by the existence of a local slice. There are algebras that are not cluster-tilted but do have a local slice.

$b = 0$ $b = 1$ $b = 2$

$g = 0$

$g = 1$

$g = 2$

Figure 5: Examples of surfaces, g is the genus and b the number of boundary components

4 Lecture 4: Cluster algebras from surfaces

Let S be an oriented Riemann surface with boundary δS, and let $M \subset S$ be a finite set of points, at least one in each boundary component. The points in $M \cap (S \setminus \delta S)$ are called *punctures*.

We will call the pair (S, M) simply a *surface*.

An *arc* is the (homotopy class) of a curve in S that connects two points in M.

An *ideal triangulation* T is a maximal set of non-crossing arcs. All triangulations of the surface have the same number of elements, which we denote by n. Examples are shown in Figure 6.

To every triangulation T, one can associate a quiver Q_T, generalizing the construction in section 3.3.

Definition 5 *A cluster algebra is said to be* from a surface *if it contains a seed* $(\mathbf{x}, \mathbf{y}, Q)$ *such that* Q *is the quiver of a triangulation of surface.*

Theorem 4.1 *[FST] For cluster algebras from surfaces*

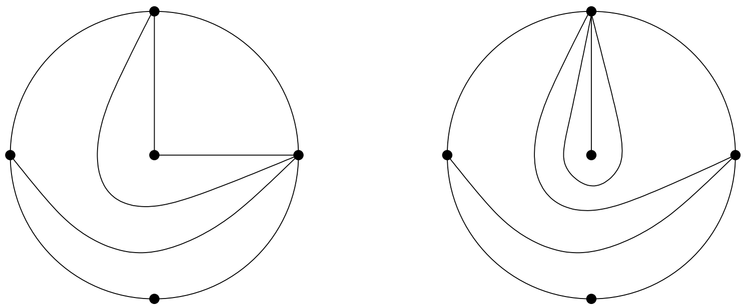


Figure 6: Two triangulations of a disc with 4 marked points on the boundary and 1 puncture. The triangulation on the right contains a self-folded triangle.

- *there are bijections*

$$\begin{array}{ccc} \{\text{arcs}\} & \longrightarrow & \{\text{cluster variables}\} \\ \gamma & \longmapsto & x_\gamma \\ & & \\ \{\text{triangulations}\} & \longrightarrow & \{\text{clusters}\} \\ T = \{\tau_1, \ldots, \tau_n\} & \longmapsto & \mathbf{x}_T = \{x_{\tau_1}, \ldots, x_{\tau_n}\} \end{array}$$

- *The triangulation $T \setminus \{\tau_k\} \cup \{\tau'_k\}$ obtained by flipping the arc τ_k (see Figure 7) corresponds to the mutation $\mu_k(\mathbf{x}_T) = \mathbf{x}_T \setminus \{x_{\tau_k}\} \cup \{x_{\tau'_k}\}$.*
- *The exchange relation corresponding to the situation in Figure 7 is*

$$x_{\tau_k} x_{\tau'_k} = *x_a x_c + *x_b x_d,$$

the product of the two diagonals equals the sum of the products of opposite sides (as before the $*$ stands for a certain coefficient).

Remark 4.2 *The last two points in the Theorem are a little more complicated if self-folded triangles are involved.*

Figure 7: Flipping of the arc τ_k

4.1 Positivity Theorem

The following theorem is the positivity conjecture for cluster algebras from surfaces.

Theorem 4.3 *[**?**] Let $\mathcal{A}$ be a cluster algebra from a surface with arbitrary coefficients. Let x_γ be any cluster variable, $\mathbf{x} = \{x_1, \cdots, x_n\}$ be any cluster, and let*

$$x_\gamma = f(\mathbf{x})/x_1^{d_1} \cdots x_n^{d_n}$$

be the Laurent expansion of x in $\mathbf{x}$, with $f \in \mathbb{Z}\mathbb{P}[\mathbf{x}]$.

Then $f \in \mathbb{Z}_{\geq 0}\mathbb{P}[\mathbf{x}]$.

The proof of the theorem is a simple consequence of an explicit expansion formula for the Laurent polynomials obtained in [**?**]. We give a sketch of this formula in the next section.

4.2 Explicit expansion formula

Let γ be an arc, T a triangulation (corresponding to the cluster $\mathbf{x}$). Suppose for simplicity that γ does not cut out a once punctured monogon.

(1) Construct a weighted graph G_γ from γ and T. The graph G_γ consists of d tiles $G_1, \ldots, G_d$, where d is the number of crossings between γ and T. A tile is a graph with 4 vertices and 4 edges that has the shape of a square.

G_γ is constructed recursively by adding one tile in each step on the upper edge or the right edge of the previous tile, see Figure 8.

Each edge of G_γ has a weight, which is either a cluster variable or 1. The weights on each tile G_i are determined by the local configuration

Figure 8: Example of a graph G_γ, not showing the weghts. The first tile the lower left tile.

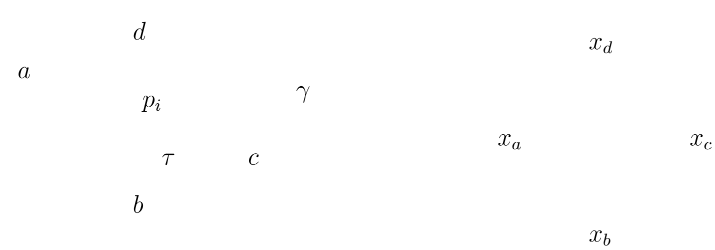


Figure 9: Weights on the tile G_i

G_γ $\quad$ P_- $\quad$ P_+ $\quad$ P $\quad$ $P_- \ominus P$

Figure 10: Examples of perfect matchings and an example of $P_- \ominus P$

at the i-th crossing point p_i between γ and T as shown in Figure 9. In the left picture, a, b, c, d, τ are arcs of the triangulation, and p_i is the i-th crossing between γ and T. The picture on the right shows the tile G_i with the cluster variables x_a, x_b, x_c, x_d as weights, here $x_e = 1$ if e is a segment of the boundary of the surface. Let $\operatorname{diag} G_i = \tau$ denote the arc in T that contains p_i.

Remark 4.4 *This is more complicated if p_j lies on a self-folded triangle.*

(2) Use perfect matchings of G_γ as summation index. Recall that a perfect matching P of a graph G is a set of edges of G such that every vertex in G is incident to exactly one edge in P.

(3) Define the products of cluster variables

 (a) $\operatorname{cross}(\gamma, T) = x_{\tau_{i_1}} \cdots x_{\tau_{i_d}}$, where $\tau_{i_1}, \dots, \tau_{i_n}$ are the arcs of T that are crossed by γ,

 (b) $x(P) = \prod_{e \in P} x_e$.

(4) Define coefficients: There are two perfect matchings P_-, P_+ that contain only boundary edges of G_γ, see Figure 10. For any matching P consider the symmetric difference $P_- \ominus P = (P_- \cup P) \setminus (P_- \cap P)$. $P_- \ominus P$ bounds a union of tiles $\cup_{j \in J} G_j$.

 Let $y(P) = \prod_{j \in J} y_{\operatorname{diag} G_j}$

Remark 4.5 *This last step is more complicated if self-folded triangles are involved, see the example in section 4.3*

(5) Then the expansion formula is

$$x_\gamma = \frac{1}{\text{cross}(\gamma, T)} \sum_P x(P)y(P)$$

where the sum is over all perfect matchings of G_γ.

The expansion formula is clearly positive. This shows the positivity theorem.

4.3 Example

The graph G_γ shown in Figure 11 has 19 perfect matchings. Because of the self-folded triangle we have to make an additional substitution $x_\ell = x_1x_2$, $x_r = x_2$ $y_\ell = y_1$, $y_r = y_2/y_1$. We also set the variables of the boundary segments equal to 1, thus $x_{11} = x_{12} = x_{13} = x_{14} = 1$.

The Laurent expansion of x_{γ_1} is equal to:

$$\begin{array}{cc}
= & \frac{1}{x_1x_2x_3x_4x_5x_6}\Big(x_1x_2x_4^2x_5x_9 \\
+ & y_3 \quad x_4x_5x_9 \\
+ & y_6 \quad x_1x_2x_4^2x_7 \\
+ & y_1y_3 \quad x_3x_4x_5x_9 \\
+ & y_3y_6 \quad x_4x_{10}x_7 \\
+ & y_5y_6 \quad x_1x_2x_4x_6x_7 \\
+ & y_2y_3 \quad x_3x_4x_5x_9 \\
+ & y_1y_3y_6 \quad x_3x_4x_{10}x_7 \\
+ & y_3y_5y_6 \quad x_6x_7 \\
+ & y_1y_2y_3 \quad x_3^2x_4x_5x_9 \\
+ & y_2y_3y_6 \quad x_3x_4x_{10}x_7 \\
+ & y_1y_3y_5y_6 \quad x_3x_6x_7 \\
+ & y_3y_4y_5y_6 \quad x_3x_5x_6x_7 \\
+ & y_1y_2y_3y_6 \quad x_3^2x_4x_{10}x_7 \\
+ & y_2y_3y_5y_6 \quad x_3x_6x_7 \\
+ & y_1y_3y_4y_5y_6 \quad x_3^2x_5x_6x_7 \\
+ & y_1y_2y_3y_5y_6 \quad x_3^2x_6x_7 \\
+ & y_2y_3y_4y_5y_6 \quad x_3^2x_5x_6x_7 \\
+ & y_1y_2y_3y_4y_5y_6 \quad x_3^3x_5x_6x_7\Big).
\end{array}$$

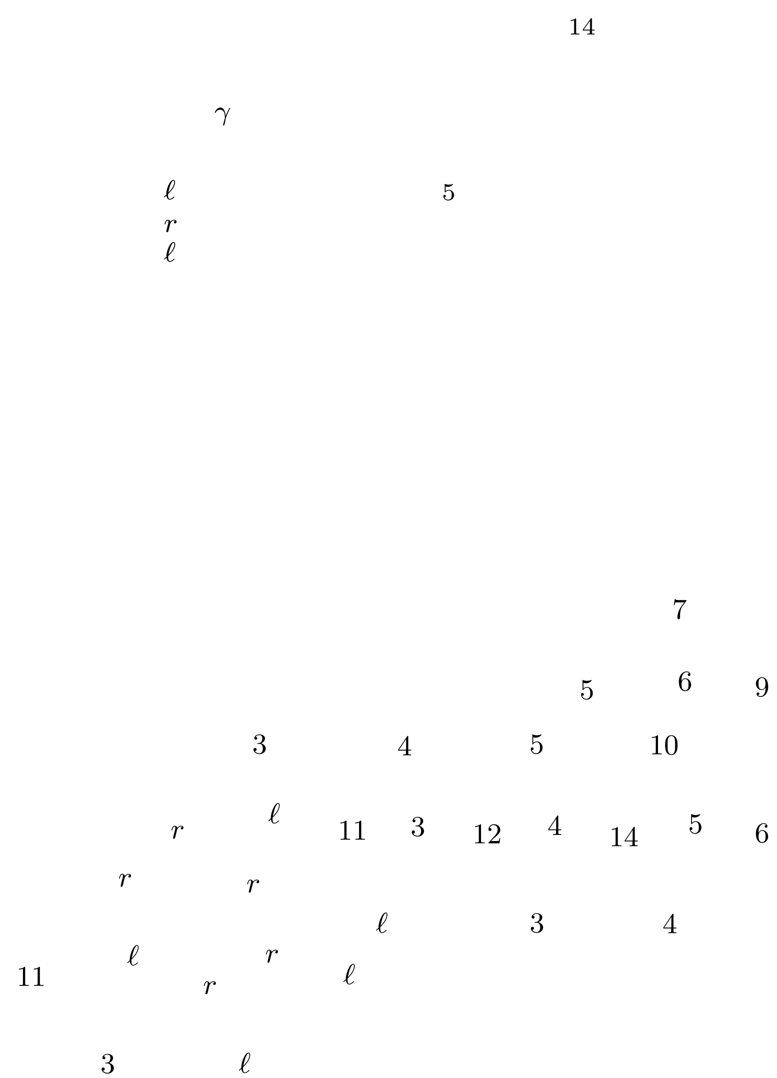


Figure 11: Triangulation T with arc γ and corresponding graph G_γ

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